Interesting Mathematical Problems to Ponder

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Exercise 1. Factor

$$a^3 + b^3 + c^3 - 3abc.$$

Solution: Consider the monic third degree polynomial whose zeros are a, b, c:

$$x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc.$$

Then

$$\begin{aligned} a^3 - (a+b+c)a^2 + (ab+bc+ca)a - abc = 0 \\ b^3 - (a+b+c)b^2 + (ab+bc+ca)b - abc = 0 \\ c^3 - (a+b+c)c^2 + (ab+bc+ca)c - abc = 0. \end{aligned}$$

Adding up these three equalities yields

$$a^3 + b^3 + c^3 - (a+b+c)(a^2+b^2+c^2) + (ab+bc+ca)(a+b+c) \\ -3abc = 0.$$

Hence

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca). \tag{1}$$

Another way to obtain the identity (1) is to consider the determinant

$$D = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

Expanding D we have

$$D = a^3 + b^3 + c^3 - 3abc.$$

On the other hand, adding up all columns yields

$$D = \begin{vmatrix} a+b+c & b & c \\ a+b+c & a & b \\ a+b+c & c & a \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & a & b \\ 1 & c & a \end{vmatrix} \\ = (a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

Note that the expression

$$a^2 + b^2 + c^2 - ab - bc - ca$$

can be also written as

$$\frac{1}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right].$$

We obtain another version of the identity (1):

$$a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a+b+c)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]. \qquad (2)$$

This form leads to a short proof of the AM-GM inequality for three variables. Indeed, from (2) it is clear that if a, b, c are nonnegative, then $a^3 + b^3 + c^3 \geq 3abc$. Now, if x, y, z are positive numbers, taking $a = \sqrt[3]{x}$, $b = \sqrt[3]{y}$, $c = \sqrt[3]{z}$ yields

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz},$$

with equality if and only if $x = y = z$.

Exercise 2. Find the minimum of $3^{x+y}(3^{x-1} + 3^{y-1} - 1)$ over all pairs (x, y) of real numbers.

Solution: Let $f(x, y) = 3^{x+y}(3^{x-1} + 3^{y-1} - 1)$. We have

$$3f(x, y) + 1 = 3^{2x+y} + 3^{x+2y} + 1 - 3 \cdot 3^{x+y},$$

which is of the form $a^3 + b^3 + c^3 - 3abc$, where $a = \sqrt[3]{3^{2x+y}}$, $b = \sqrt[3]{3^{x+2y}}$, and $c = 1$ are all positive real numbers. From (2) it follows that $3f(x, y) + 1 \geq 0$ for all $x, y \in \mathbb{R}$, with equality if and only if $x = y = 0$. Hence the minimum of $f(x, y)$ is $-\frac{1}{3}$.

The same conclusion follows directly from the AM-GM inequality, because

$$3^{2x+y-1} + 3^{x+2y-1} + 3^{-1} \geq 3\sqrt[3]{3^{2x+y-1+x+2y-1-1}} = 3^{x+y},$$

implying

$$3^{2x+y-1} + 3^{x+2y-1} - 3^{x+y} \geq -\frac{1}{3}.$$

Hence

$$3^{x+y}(3^{x-1} + 3^{y-1} - 1) \geq -\frac{1}{3},$$

for all real numbers x, y, with equality if and only if $2x + y - 1 = x + 2y - 1 = -1$, i.e. $x = y = 0$.

Exercise 3. If $a+b+c=0$, then $a^3+b^3+c^3=3abc$.

Solution: Follows immediately from (2).

Problem 1. Simplify

$$(x+2y-3z)^3+(y+2z-3x)^3+(z+2x-3y)^3.$$

Solution: Setting $x+2y-3z=a$, $y+2z-3x=b$, $z+2x-3y=c$, we have $a+b+c=0$, and from Exercise 2 it follows that $a^3+b^3+c^3=3abc$. Hence the given expression is equal to

$$3(x+2y-3z)(y+2z-3x)(z+2x-3y).$$

Problem 2. Let a,b,c be complex numbers. Prove that $a^2b+b^2c+c^2a=ab^2+bc^2+ca^2$ if and only if $a=b, orb=c$, or $c=a$.

Solution: Because $(a-b)+(b-c)+(c-a)=0$,

$$(a-b)^3+(b-c)^3+(c-a)^3=3(a-b)(b-c)(c-a),$$

so assuming $a^2b+b^2c+c^2a=ab^2+bc^2+ca^2$ yields

$$\begin{aligned}3(a-b)(b-c)(c-a)=a^3-b^3+b^3-c^3+c^3-a^3-3[(a^2b+b^2c+c^2a)\\ -(ab^2+bc^2+ca^2)]=0.\end{aligned}$$

Then $a=b$, or $b=c$, or $c=a$. The converse follows immediately.

The conclusion of the problem follows directly from

$$\begin{aligned}(a-b)(b-c)(c-a)&=abc-(a^2b+b^2c+c^2a)+(ab^2+bc^2+ca^2)]-abc\\ &=(ab^2+bc^2+ca^2)-(a^2b+b^2c+c^2a).\end{aligned}$$

Problem 3. Let x,y,z be distinct real numbers. Prove that

$$\sqrt[3]{x-y}+\sqrt[3]{y-z}+\sqrt[3]{z-x}\neq 0.$$

Solution: Assume the contrary, and let $\sqrt[3]{x-y}=a$, $\sqrt[3]{y-z}=b$, $\sqrt[3]{z-x}=c$. Then $a+b+c=0$, and, from Exercise 2, $a^3+b^3+c^3=3abc$. This yields

$$0=(x-y)+(y-z)+(z-x)=3\sqrt[3]{x-y}\sqrt[3]{y-z}\sqrt[3]{z-x}\neq 0,$$

a contradiction. The problem is solved.

Problem 4. Let r be a real number such that $\sqrt[3]{r}-\frac{1}{\sqrt[3]{r}}=2$. Find $r^3-\frac{1}{r^3}$. (UWW Mathmeet, 2003)

Solution: With $a=\sqrt[3]{r}$, $b=-\frac{1}{\sqrt[3]{r}}$, $c=-2$, we have again $a+b+c=0$, hence $a^3+b^3+c^3=3abc$. This yields

$$r-\frac{1}{r}-8=3\sqrt[3]{r}\left(-\frac{1}{\sqrt[3]{r}}\right)(-2),$$

or, equivalently,

$$r - \frac{1}{r} - 14 = 0.$$

By applying the result in Exercise 2 again, we get

$$r^3 - \frac{1}{r^3} - 2744 = 3r\left(-\frac{1}{r}\right)(-14),$$

or

$$r^3 - \frac{1}{r^3} = 2744 + 42 = 2786.$$

Problem 5. Show that if the numbers $\overline{abc}$, $\overline{bca}$, $\overline{cab}$ are divisible by n, then so is $a^3 + b^3 + c^3 - 3abc$.

Solution: We have seen that

$$a^3 + b^3 + c^3 - 3abc = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = \begin{vmatrix} 100b + 10c + a & b & c \\ 100a + 10b + c & a & b \\ 100c + 10a + b & c & a \end{vmatrix}$$
$$= \begin{vmatrix} \overline{bca} & b & c \\ \overline{abc} & a & b \\ \overline{cab} & c & a \end{vmatrix},$$

and the conclusion follows.

Problem 6. The number of ordered pairs of integers (m, n) such that $mn \geq 0$ and $m^3 + 99mn + n^3 = 33^3$ is
a) 2 b) 3 c) 33 d) 35 e) 99.
(AHSME 1999)

Solution: Write the given relation as

$$m^3 + n^3 + (-33)^3 - 3mn(-33) = 0.$$

From the identity (2) it follows that

$$(m + n - 33)\left[(m - n)^2 + (m + 33)^2 + (n + 33)^2\right] = 0.$$

The equation $m + n = 33$, along with the condition $mn \geq 0$, yields 34 solutions: $(k, 33 - k)$, $k = 0, 1 \ldots, 33$. The second factor is equal to zero only when $m = n = -33$, giving the 35th solution.